

BAB V

PERSAMAAN DIFERENSIAL PARSIAL

Pendahuluan

Persamaan diferensial parsial adalah persamaan yang memuat satu atau lebih turunan parsial dengan dua atau lebih variabel bebas. Orde dari PD parsial : tingkat tertinggi dari derivatif yang ada dalam PD. Derajat dari PD parsial : pangkat tertinggi dari turunan tingkat tertinggi yang ada dalam PD.

PD parsial dikatakan **linier** jika hanya memuat derajat pertama dari variabel - variabel bebasnya dan derivatif - derivatif parsialnya. Beberapa contoh PD parsial yang penting :

1. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ persamaan gelombang satu dimensi
2. $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ persamaan konduksi panas satu dimensi
3. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ persamaan laplace dua dimensi
4. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$ persamaan poisson dua dimensi
5. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ persamaan laplace tiga dimensi

CONTOH-CONTOH PD PARSIAL DALAM BENTUK PRAKTIS:

- 1. Persamaan Gelombang : $\frac{\partial^2 U}{dt^2} = C^2 \frac{\partial^2 U}{dx^2}$
- 2. Persamaan Gelombang Radio : $-\frac{\partial V}{dx} = L \frac{\partial U}{dt}$
 $-\frac{\partial I}{dx} = C \frac{\partial V}{dt}$
- 3. Persamaan Panas (heat flow)
 untuk dimensi satu : $\frac{\partial U}{dt} = C^2 \frac{\partial^2 U}{dx^2}$
- 4. Persamaan Panas (heat flow)
 untuk dimensi dua : $\frac{\partial^2 U}{dx^2} + \frac{\partial^2 U}{dy^2} = 0$

Pembentukan PD Parsial

Membentuk persamaan differensial parsial dapat dilakukan dengan :

- A. Eliminasi konstanta
- B. Eliminasi fungsi.

A. Eliminasi konstanta

Contoh :

Bentuklah PD parsial dari : $x^2 + y^2 + (z - c)^2 = a^2$

Jawab:

$$x^2 + y^2 + (z - c)^2 = a^2 \text{ (Ada 2 konstanta yaitu } a \text{ dan } c)$$

Turunkan persamaan terhadap x: $2x + 2(z - c) \frac{\partial z}{\partial x} = 0$

$$x + (z - c) \frac{\partial z}{\partial x} = 0 \dots \dots \dots (1)$$

Turunkan persamaan terhadap y : $2y + 2(z - c) \frac{\partial z}{\partial y} = 0$

$$y + (z - c) \frac{\partial z}{\partial y} = 0 \dots \dots \dots (2)$$

Eliminasi c dengan cara: $[(1) \times \frac{\partial z}{\partial y} - (2) \times \frac{\partial z}{\partial x}]$ sehingga:

$$x \frac{\partial z}{\partial y} - y \frac{\partial z}{\partial x} = 0$$

B. Eliminasi fungsi

Contoh: Bentuklah PD Parsial dari: $z = f(x^2 - y^2)$

Jawab : $p = \frac{\partial z}{\partial x} = f'(x^2 - y^2)(2x) \dots \dots \dots (1)$

$$q = \frac{\partial z}{\partial y} = f'(x^2 - y^2)(-2y) \dots \dots \dots (2)$$

dari (1) dan (2) didapat :

$$\frac{p}{q} = \frac{2x}{-2y} \leftrightarrow 2xq + 2yp = 0$$

sehingga:

$$x \frac{\partial z}{\partial y} + y \frac{\partial z}{\partial x} = 0$$

PD Parsial Linier Orde 2

Persamaan umum :

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = G \dots \dots \dots (5-1)$$

u = variabel tak bebas, merupakan fungsi dari x dan y

x, y = variabel bebas dari PD

A, B, C, D, E, F, G = koefisien, bisa konstan atau merupakan fungsi dari x atau y tetapi bukan fungsi dari u.

Jika: $G = 0$ disebut PD homogen

$G \neq 0$ disebut PD non homogen

Jika: $B^2 - 4ac < 0$ disebut PD Eliptik

$B^2 - 4ac = 0$ disebut PD Parabolis

$B^2 - 4ac > 0$ disebut PD Hiperbolis

Metode Penyelesaian PD Parsial

Beberapa Penyelesaian PD parsial yang akan dibahas adalah:

- A. Integral Langsung
- B. Pemisalan $u = e^{ax+by}$
- C. Pemisahan Variabel

Integral Langsung

Mencari penyelesaian umum dengan metoda yang digunakan dalam PD biasa (dengan mengintegalkan masing - masing ruas ke setiap variabel bebasnya).

Contoh :

a. Selesaikan PD : $\frac{\partial^2 z}{\partial x \partial y} = x^2 y$

b. Tentukan masalah nilai batas yang memenuhi $z(x, 0) = x^2$; $z(1, y) = \cos y$

PENYELESAIAN :

a. $\frac{\partial^2 z}{\partial x \partial y} = x^2 y$

$$\frac{\partial}{\partial x} \left[\frac{\partial z}{\partial y} \right] = x^2 y$$

→ Diintegalkan terhadap x

$$\frac{\partial z}{\partial y} = \frac{1}{3} x^3 y + F(y)$$

→ Diintegalkan terhadap y

$$z = \frac{1}{3} x^3 y^2 + \int F(y) dy + G(x)$$

PUPD : ; $z = \frac{1}{6} x^3 y^2 + H(y) + G(x)$ $G(x)$ dan $H(y)$ fungsi sembarang

b. $z(x,0) = x^2 \rightarrow x^2 = \frac{1}{6} x^3 0^2 + H(0) + G(x)$

$$G(x) = x^2 - H(0)$$

$$z(1, y) = \cos y \rightarrow \cos y = \frac{1}{6} 1^3 y^2 + H(y) + 1^2 - H(0)$$

$$z(x, y) = \frac{1}{6} x^3 y^2 + H(y) + x^2 - H(0)$$

$$\cos y = \frac{1}{6} y^2 + H(y) + 1^2 - H(0)$$

$$H(y) = \cos y - \frac{1}{6} y^2 - 1 + H(0)$$

$$z(x, y) = \frac{1}{6} x^3 y^2 + \cos y - \frac{1}{6} y^2 - 1 + H(0) + x^2 - H(0)$$

$$\text{PKPD : } z(x, y) = \frac{1}{6} x^3 y^2 + \cos y - \frac{1}{6} y^2 - 1 + x^2$$

$$2. \text{ Selesaikan PD : } t \frac{\partial^2 u}{\partial x \partial t} + 2 \frac{\partial u}{\partial x} = x^2 \quad ; \quad u(x, 1) = \frac{x^3}{6}; u(0, t) = 0$$

$$\frac{\partial}{\partial x} \left[t \frac{\partial u}{\partial t} + 2u \right] = x^2 \quad \longrightarrow \text{ diintegrasikan ke } x$$

$$t \frac{\partial u}{\partial t} + 2u = \frac{1}{3} x^3 + F(t) \quad \longrightarrow \text{ dikalikan } t$$

$$t^2 \frac{\partial u}{\partial t} + 2tu = \frac{1}{3} x^3 t + tF(t)$$

$$\frac{\partial}{\partial t} (t^2 u) = \frac{1}{3} x^3 t + tF(t) \quad \longrightarrow \text{ diintegrasikan ke } t$$

$$t^2 u = \frac{1}{6} x^3 t^2 + \int tF(t) dt + H(x)$$

$$t^2 u = \frac{1}{6} x^3 t^2 + G(t) + H(x)$$

$$\text{PUPD: } u(x, t) = \frac{\left[\frac{1}{6} x^3 t^2 + G(t) + H(x) \right]}{t^2}$$

$$\text{Syarat batas 1 : } u(x, 1) = \frac{x^3}{6}$$

$$u(x, 1) = \frac{\left[\frac{1}{6} x^3 + G(1) + H(x) \right]}{1} = \frac{1}{6} x^3$$

$$\frac{1}{6}x^3 + G(1) + H(x) = \frac{1}{6}x^3 \rightarrow G(1) + H(x) = 0 \rightarrow H(x) = -G(1)$$

Penyelesaian :
$$u(x, t) = \frac{\left[\frac{1}{6}x^3 t^2 + G(t) - G(1) \right]}{t^2}$$

Syarat batas 2 : $u(0, t) = 0$

$$u(0, t) = \frac{0 + G(t) - G(1)}{t^2} = 0$$

$$G(t) - G(1) = 0$$

$$G(t) = G(1)$$

$$u(x, t) = \frac{\frac{1}{6}x^3 t^2 + G(1) - G(1)}{t^2}$$

$$\text{PKPD : } u(x, t) = \frac{1}{6}x^3$$

Pemisalan $u = e^{ax+by}$

PD parsial linear orde 2 dengan A,B,C,D,E,F konstan, PU PD ditentukan dengan memisalkan $u = e^{ax+by}$; a,b konstanta yang harus dicari.

Contoh:

1. Selesaikan PD : $3 \frac{\partial u}{\partial x} + 2 \frac{\partial u}{\partial y} = 0$; $u(x, 0) = 4e^{-x}$

Penyelesaian :

misalkan : $u(x, y) = e^{ax+by}$

$$\frac{\partial u}{\partial x} + ae^{ax+by} ; \quad \frac{\partial u}{\partial y} = be^{ax+by}$$

PD menjadi :

$$3ae^{ax+by} + 2be^{ax+by} = 0$$

$$(3a + 2b)e^{ax+by} = 0 \quad \longrightarrow \quad 3a + 2b = 0 \rightarrow b = -\frac{3}{2}a$$

$$\text{PU PD : } u(x, y) = e^{ax - \frac{3}{2}ay} = e^{a\left(x - \frac{3}{2}y\right)} = F\left(x - \frac{3}{2}y\right)$$

syarat batas : $u(x, 0) = 4e^{-x}$

$$u(x, 0) = F(x) = 4e^{-x}$$

2. PD : $\frac{\partial v}{\partial x} + 3 \frac{\partial v}{\partial y} = 0$, dengan syarat batas : $v(0, y) = 4 \sin y$

misalkan $v(x, y) = e^{ax+by}$

$$\frac{\partial v}{\partial x} = ae^{ax+by} \quad ; \quad \frac{\partial v}{\partial y} = be^{ax+by}$$

PD menjadi :

penyelesaian PD : $u(x, y) = 4e^{-\left(x-\frac{3}{2}y\right)} = 4e^{(3y-2x)/2}$
 $(a + 3b)e^{ax+by} = 0 \longrightarrow a + 3b = 0 \rightarrow a = -3b$

PU PD :

$$v(x, y) = e^{-3bx+by} = e^{b(-3x+y)} = F(-3x + y)$$

syarat batas: $v(0, y) = 4 \sin y$

$$v(0, y) = F(y) = 4 \sin y$$

penyelesaian PD : $v(x, y) = 4 \sin(-3x + y)$

3. Selesaikan PD : $\frac{\partial^2 u}{\partial x^2} + 3 \frac{\partial^2 u}{\partial x \partial y} + 2 \frac{\partial^2 u}{\partial y^2} = 0$; $u(0, y) = y$; $u_x(0, y) = 0$

Penyelesaian :

misalkan : $u = e^{ax+by}$

$$\frac{\partial u}{\partial x} = ae^{ax+by} \quad \frac{\partial^2 u}{\partial x^2} = a^2 e^{ax+by}$$

$$\frac{\partial u}{\partial y} = be^{ax+by} \quad \frac{\partial^2 u}{\partial y^2} = b^2 e^{ax+by}$$

$$\frac{\partial^2 u}{\partial x \partial y} = abe^{ax+by} = \frac{\partial^2 u}{\partial y \partial x}$$

PD menjadi:

$$(a^2 + 3ab + 2b^2) e^{ax+by} = 0$$

$$a^2 + 2ab + 2b^2 = 0$$

$$(a + b)(a + 2b) = 0 \rightarrow a = -b \text{ atau } a = -2b$$

$$\text{untuk } a = -b \rightarrow u_1 = e^{-bx+by} = e^{b(-x+y)} \rightarrow u_1 = F(-x + y)$$

$$\text{Untuk } a = -2b \rightarrow u_2 = e^{-2bx+by} = e^{b(-2x+y)} \rightarrow u_2 = G(-2x + y)$$

$$\text{PU PD : } \begin{aligned} u(x, y) &= u_1 + u_2 \\ u(x, y) &= F(-x + y) + G(-2x + y) \end{aligned}$$

$$u(0, y) = y$$

$$\text{Syarat batas 1 : } u(0, y) = f(y) + G(y) = y$$

$$\rightarrow G(y) = -F(y) + y$$

$$\text{Penyelesaian PD : } u(x, y) = F(-x + y) - \frac{F(-2x + y) - 2x + y}{G(-2x + y)}$$

$$\text{Syarat batas 2 : } u_x(0, y) = 0$$

$$\text{Misalkan ; } -x + y = v \text{ dan } -2x + y = w$$

$$u = F(v) - F(w) + w$$

$$u_x(x, y) = \frac{\partial u(x, y)}{\partial x} = \frac{dF}{dv} \frac{\partial v}{\partial x} - \frac{dF}{dw} \frac{\partial w}{\partial x} - \frac{\partial w}{\partial x}$$

$$= \frac{dF}{dv}(-1) - \frac{dF}{dw}(-2) - 2 = -\frac{dF}{dv} + 2\frac{dF}{dw} - 2$$

$$= -F'(v) + 2F'(w) - 2$$

$$= -F'(-x + y) + 2F'(-2x + y) - 2$$

$$u_x(0, y) = -F'(y) + 2F'(y) - 2 = 0$$

$$\rightarrow F'(y) = 2$$

$$\rightarrow F(y) = 2y + c$$

$$u(x, y) = f(-x + y) - F(-2x + y) - 2x + y$$

$$u(x, y) = 2(-x + y) + c - [2(-2x + y) + c] - 2x + y$$

$$u(x, y) = -2x + 2y + c + 4x - 2y - c - 2x + y$$

$$u(x, y) = y$$

Pemisahan Variabel: $u = x \cdot y$ dengan $x = x(x)$, $y = y(y)$, $\frac{\partial u}{\partial x} = x'y$, $\frac{\partial u}{\partial y} = xy'$

B. Metode Pemisahan Variabel $U = X Y$, $X = X(x)$, $Y = Y(y)$

$$\frac{\partial U}{\partial x} = X' Y ; \quad \frac{\partial U}{\partial y} = X Y'$$

Contoh:

1. Selesaikan PD Parsial $\frac{\partial U}{\partial x} = 4 \frac{\partial U}{\partial y}$, $U(0,y) = 8 e^{-3y}$

Jawab :

$$\text{Misal } U = X Y \rightarrow \frac{\partial U}{\partial x} = X' Y ; \quad \frac{\partial U}{\partial y} = X Y'$$

Jadi PDP menjadi $X' Y = 4 X Y'$ atau

$$\frac{X'}{X} = 4 \frac{Y'}{Y} = k \quad (\text{konstanta})$$

$$\frac{X'}{X} = k \quad \text{dan} \quad 4 \frac{Y'}{Y} = k$$

$$X = C_1 e^{kX} \quad \text{dan} \quad Y = C_2 e^{\frac{k}{4}Y}, \text{ sehingga:}$$

$$U(x,y) = U = X Y = C_1 e^{kX} \cdot C_2 e^{\frac{k}{4}Y} = C e^{kx + \frac{k}{4}y}, \quad C = C_1 C_2$$

$$\text{Karena } U(0,y) = 8 e^{-3y} = C e^{0 + \frac{k}{4}y} = C e^{\frac{k}{4}y}, \text{ maka}$$

$$\text{Dari identitas diperoleh : } C = 8, \quad k/4 = -3 \rightarrow k = -12$$

$$\text{Jadi Penyelesaian PDP adalah } U(x,y) = 8 e^{-12x - 3y} //$$

2. Selesaikan PD Parsial $\frac{\partial U}{\partial x} + U = \frac{\partial U}{\partial t}$, $U(x,0) = 4 e^{-3x}$

Jawab :

$$\text{Misal } U = X T \rightarrow \frac{\partial U}{\partial x} = X^1 T ; \quad \frac{\partial U}{\partial t} = X T^1$$

Jadi PDP menjadi $X^1 T + X T = X T^1$ atau

$$\frac{X^1}{X} + 1 = \frac{T^1}{T} = k \quad (\text{konstanta})$$

$$\frac{X^1}{X} + 1 = k \quad \text{dan} \quad \frac{T^1}{T} = k$$

$X = C_1 e^{(k-1)X}$ dan $Y = C_2 e^{kT}$, sehingga:

$$U(x,t) = U = X T = C_1 e^{(k-1)X} \cdot C_2 e^{kT} = C e^{(k-1)X + kT} , \quad C = C_1 C_2$$

Karena $U(x,0) = 4 e^{-3x} = C e^{(k-1)X + 0} = C e^{(k-1)X}$, maka

Dari identitas diperoleh : $C = 4$, $k - 1 = -3 \rightarrow k = -2$

Jadi Penyelesaian PDP adalah $U(x,y) = 4 e^{-3x-2t}$ //

SOAL-SOAL LATIHAN

1. Selesaikan PD Parsial $4 \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 3 u$, $u(0,x) = 3 e^{-x} - e^{-5x}$

2. Selesaikan PD Parsial $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$

3. Selesaikan PD Parsial $\frac{\partial u}{\partial x} - 2 \frac{\partial u}{\partial y} = u$; syarat : $u(x,0) = 3 e^{-5x} + 2 e^{-3x}$

4. Selesaikan PD Parsial $4 \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$, $u(0,y) = e^{-5y}$